

Introduction to the
Yang-Mills equations on
Riemann surfaces.

Notes from an 8-hour course given
at USTC, Hefei, P.R.C. in July 2015.

The author thanks Prof. Xiuxiang Chen
for the opportunity to deliver these four lectures.

Alex Waldron

References:

1. Doi's H. 2. Atiyah - Bott

3. Durr-Klein

4. K.T. G. : Lecture on Hamiltonian. Einstein constraints...

1

Vector bundles

Let $\pi: E \rightarrow M$ be a submersion.

a section $s: U \rightarrow E$ over an open set $U \subset M$ is a smooth map s.t.

$$\pi \circ s = \text{Id}_U.$$

$$\Leftrightarrow s(x) \in \pi^{-1}(x) \quad \forall x \in U.$$

We say E is a $\mathbb{C}^\infty$ vector bundle if:

(i) Each fiber $\pi^{-1}(x)$ has the structure of a vector space.

(ii) Each point $x \in M$ has a nbhd $U \ni x$

and a local frame $\{e_i\}_{i=1}^n$ of sections over U .

any section $s \in \Gamma(U; E)$

can be written $s = \sum s^\alpha e_\alpha$

[Like basis.]

$$s = \sum_{\alpha=1}^n s^\alpha e_\alpha$$

Einstein summation convention.

for unique smooth functions s^α .

= local components of s .

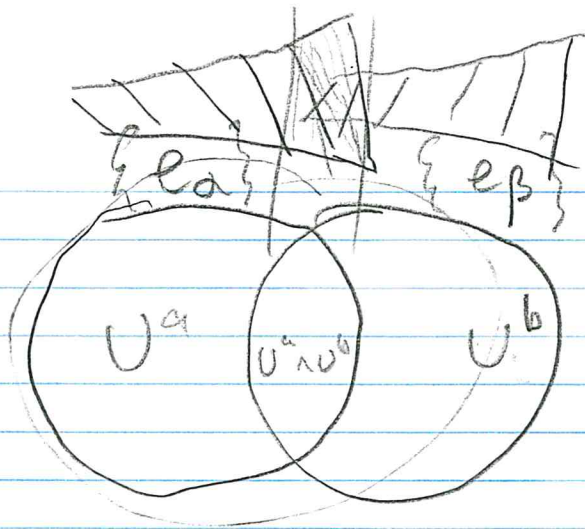
Note: Every vector bundle is locally trivial.

The map $U \times \mathbb{R}^n \rightarrow \pi^{-1}(U) \subset E$

$$(x, y^1, \dots, y^n) \mapsto y^1 e_1(x) + \dots + y^n e_n(x) \in \pi^{-1}(x)$$

is an isomorphism.

$\hookrightarrow$ $\mathbb{R}^n$ -vector space from an each fiber.



$$e_\alpha = e_\alpha^a$$

$$e_\beta = e_\beta^b$$

Define the transition functions
over $U^a \cap U^b$ (by (ii))

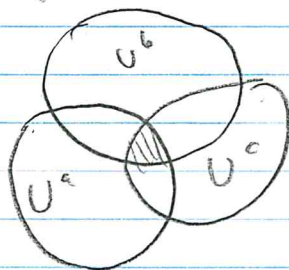
$$e_\alpha = u^\beta \times e_\beta$$

For any section $s \in T(U^a \cap U^b, E)$,

$$\begin{aligned} s &= s^\alpha e_\alpha = s^\alpha (u^\beta \times e_\beta) \\ &= (u^\beta \times s^\alpha) e_\beta = s^\beta e_\beta \end{aligned}$$

$$\Rightarrow \boxed{s^\beta = u^\beta \times s^\alpha}$$

"transformation rule":



{e_\alpha} {e_\beta} {e_\gamma}

Over $U^a \cap U^b \cap U^c$, there holds

$$u^\gamma \times s^\alpha = s^\beta = u^\beta s^\beta = u^\beta u^\alpha \times s^\alpha$$

$$\Rightarrow \underline{u^\gamma_\alpha = u^\beta_\alpha u^\alpha_\beta} \quad (\text{Cocycle condition.})$$

$$u^\beta_\alpha = (u^{\alpha\beta})^\beta_\alpha$$

Conversely, any collection [of invertible matrix-valued functions] $\{n^{a \times b}\}$ over $U^a \cap U^b$, which satisfy the cocycle conditions, define a vector bundle over M !

[equivalence relation!]
Classification of C^∞ vector bundles:
Milnor & Stasheff: Characteristic classes
Bott & Tu: Differential forms in Algebraic Topology.

Any natural operation on vector spaces
 $\oplus \quad \otimes \quad (-)^* \quad \wedge^k$

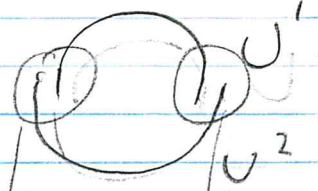
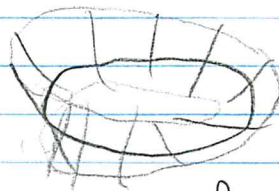
can be performed on vector bundles. [do on local triv's & glue.]

$\{e_+\}$ - local frame for E
 $\Rightarrow \{e_+^*\}$ - dual frame for E^*
 sections of $E^* \rightsquigarrow S \otimes e_+^*$
 sections of $E \otimes E^* = \text{End } E$

$\rightsquigarrow n^{\mathbb{R}_2} \alpha_1 \dots \alpha_n$ [sum to transform]

Examples:

- Möbius strip!
 $M = S^1$
 line bundle.



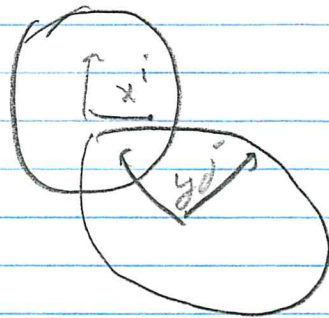
$n^{\mathbb{R}_2} \alpha = +1 \quad -1$

(More generally) but doing constant action: $M = S^m$
 (valued in $GL(n)$)

- Tangent bundle TM

$\{x^i\}_{i=1}^m$ local coords, $m = \dim M$

$\left\{ \partial_i = \frac{\partial}{\partial x^i} \right\}_{i=1}^m$ local frame for $T\mathcal{M}$.



$$\frac{\partial}{\partial x^i} = \left(\frac{\partial y^j}{\partial x^i} \right) \frac{\partial}{\partial y^j}$$

transition func = Jacobian matrices.

- $T^*M = \text{cotangent bundle} = (TM)^*$

$$dx^i = \left(\frac{\partial x^i}{\partial y^j} \right) dy^j$$

transpose.

- $\Omega^k = \Lambda^k T^*M = \text{bundle of } k\text{-forms}$
frame: $\{dx^{i_1} \wedge \dots \wedge dx^{i_k}\}$.

[The last 3 have special structure.
most do not.]

a metric $\langle \cdot, \cdot \rangle$ on E is a smoothly varying inner-product on each fiber $\pi^{-1}(x)$.

$$\langle s, t \rangle = s^\alpha h_{\alpha\beta} t^\beta \in \mathbb{R},$$

$\Rightarrow (h_{\alpha\beta})$ is a section of $S^2(E^*) \subset E^* \otimes E^*$

By the Gram-Schmidt process,
 it is ^{always} possible to choose each local
 frame $\{e_x\}$ to be orthogonal.

$$\begin{aligned} \delta_{x_1, x_2} &= \langle e_{x_1}, e_{x_2} \rangle \\ &= \langle n^{\beta_1}_{x_1} e_{\beta_1}, n^{\beta_2}_{x_2} e_{\beta_2} \rangle \\ &= n^{\beta_1}_{x_1} n^{\beta_2}_{x_2} \delta_{\beta_1, \beta_2} \end{aligned}$$

$$\Rightarrow n^{\beta}_{x_2} \in O(n).$$

In general, if $n^{\beta}_{x_2} \in G \in M^{n \times n}$,
 we say E has structure group G .

Define the set of Gauge Transformations

$$g_E \subset E \otimes E^* \text{ s.t. } \pi^{-1}(x) \cap g_E = G.$$

$$\begin{array}{c} E \xrightarrow{u} E \\ \downarrow n \end{array}$$

$\leadsto$ automorphisms of E . [well-defined!]

Note: $\text{def } G \subset O(n)$, then g_E preserves $\langle \cdot, \cdot \rangle$ (orthogonal unitaries).

Let $\mathfrak{g} = \text{Lie}(G)$ and define [usually the case!]

$$g_E \subset E \otimes E^* \text{ s.t. } \pi^{-1}(x) \cap g_E = \mathfrak{g}.$$

$\leadsto$ infinitesimal gauge transformations

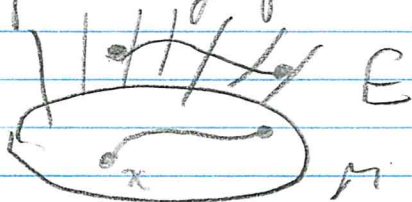
(shew - symmetric matrices)

$$s \in \mathfrak{g}_E \Rightarrow \exp(s) \in g_E.$$

[Explain: Gauge = frame, gauge trans. = change of frame.]
 This works to what: coord. basis, orthonormal

Connections and Covariant Derivatives

Informally: a connection, A , is a ^{smooth} metric-preserving rule for transporting points in the fibers of E .



Formally: a covariant derivative $\Rightarrow$ connection is a map

$$T(E) \longrightarrow T(T^*M \otimes E)$$

$$s \longmapsto \nabla_A s.$$

satisfying

$$(i) \nabla_A(f \cdot s) = df \otimes s + f \nabla_A s \quad \forall f \in C^\infty$$

$$(ii) d\langle s, t \rangle = \langle \nabla_A s, t \rangle + \langle s, \nabla_A t \rangle.$$

In local coordinates, we write

$\{e_\alpha\} = \{e_\mu\}$

$$\nabla_A e_\alpha = A_i^\beta \partial x^i \otimes e_\beta$$

to define the connection matrices A_i^β .

$$(ii) \Rightarrow 0 = d\langle e_\alpha, e_\beta \rangle = \langle A_i^\alpha \partial x^i e_\alpha, e_\beta \rangle + \langle e_\alpha, A_i^\beta \partial x^i e_\beta \rangle$$

$$0 = A_i^\beta \partial x^i e_\alpha + A_i^\alpha \partial x^i e_\beta$$

So: $\{e_\alpha\}$ orthogonal $\Rightarrow A_i$ skew-symmetric. (if $A_i^\alpha \partial x^i e_\alpha$ we say A is compatible w/ G)

$$s = s^\alpha e_\alpha \Rightarrow \nabla_A s = \partial_i s^\alpha \partial x^i \otimes e_\alpha + s^\alpha A_i^\beta \partial x^i \otimes e_\beta$$

$$= (\partial_i s^\alpha + A_i^\alpha{}_\beta s^\beta) \partial x^i \otimes e_\alpha$$

$$= (\nabla_A s)^\alpha \partial x^i \otimes e_\alpha =: \nabla_i s^\alpha.$$

How does $A: \beta$ transform?

$$u \in \mathcal{T}_E$$

$$u(\nabla_A s) = \nabla_{u(A)} u(s) \quad \forall s \in T^*(E).$$

$$\Leftrightarrow u(\nabla_A(u^{-1}(s))) = \nabla_{u(A)} s \quad \forall s.$$

$$\left(\begin{aligned} \text{Recall } 0 &= d(uu^{-1}) = du \cdot u^{-1} + u d(u^{-1}) \\ \Rightarrow -u^{-1} \cdot du \cdot u^{-1} &= d(u^{-1}). \end{aligned} \right)$$

$$\Rightarrow u \cdot (1 - u^{-1} du u^{-1}) + A \cdot u^{-1} = u(A).$$

$$\boxed{u(A) = u A u^{-1} - du \cdot u^{-1}}$$

Check: $u(A): \beta = u^x A: \delta u^{-1} \delta \beta - \delta u^x \delta u^{-1} \delta \beta$

In orthogonal frame: $u^{-1} = u^T$

Note: a connection is not a tensor!

However, the difference of two connections is a $\mathfrak{g}_E$ -valued 1-form.

$$\begin{aligned} u(B) - u(A) &= u B u^{-1} - du u^{-1} \\ &\quad - u A u^{-1} + du u^{-1} \\ &= u(B - A) u^{-1} \quad \checkmark \end{aligned}$$

$\therefore$ the space of connections A_E is an affine space modelled on $\Omega^1(\mathfrak{g}_E)$.

- any $A: \beta$
- Abelian
- Levi-Civita

How does a connection work, by itself?

8

Parallel transportation.

We say a section $s \in T(E)$ is parallel along a path $\gamma(t)$ if

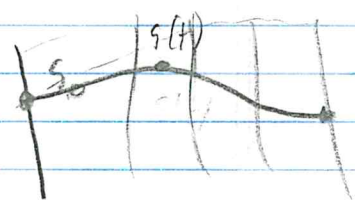
$$\frac{D\gamma(t)^i}{dt} \nabla_i s(\gamma(t)) = 0.$$

$$\boxed{\frac{ds^\alpha}{dt} = - \frac{d\gamma^i}{dt} A_i^\alpha{}_\beta s^\beta.}$$

Linear ODE

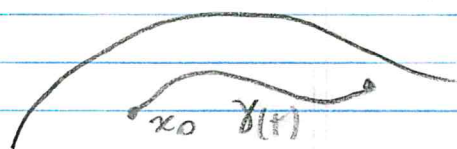
$$\frac{d}{ds} (1 + s A(s)) e_0 = A(s) e_0 = A(s)$$

$\Rightarrow$ for any $s_0 \in \pi^{-1}(x_0)$



There exists a solution

$T_{\gamma(t)} s_0$ over $\gamma(t)$.



= parallel transport of s_0 along $\gamma(t)$.

cf $A_i \in \mathfrak{g}$, then $T_{\gamma(t)} s_0 \in G \cdot s_0$.

Exercises:

1. Show that any skew-symmetric matrices

A_i define an $O(n)$ -connection, locally.

2. Use a partition of unity to show that any vector bundle has a connection, globally.

3. Let g_{ij} be a metric on TM . Show that the Levi-Civita conn. is the unique g -compatible conn. with $T_i^j = T_i^j$ (torsion-free.)

Curvature.

[First I will define it & derive its properties in the most efficient way, then say what it means.]

Given a connection A on E , we define a connection on E^* by

$$\nabla_i s_\alpha = \partial_i s_\alpha - A_i^\beta{}_\alpha s_\beta$$

and on $E \otimes E^*$:

$$\nabla_i u^\alpha s_\alpha = \partial_i u^\alpha s_\alpha + A_i^\alpha{}_\beta u^\beta s_\alpha - A_i^\alpha{}_\beta u^\alpha s_\beta$$

$$\Rightarrow \nabla T u = T \nabla u \Rightarrow \nabla T = 0.$$

Also define the covariant differential

$$D_A: T(\Omega^k(E)) \rightarrow T(\Omega^{k+1}(E))$$

$$D_A(s^\alpha dx^{i_1} \wedge \dots \wedge dx^{i_k} \otimes e_\alpha)$$

$$= (\nabla_i s^\alpha dx^i) \wedge dx^{i_1} \wedge \dots \wedge dx^{i_k} \otimes e_\alpha - s^\alpha dx^{i_1} \wedge \dots \wedge dx^{i_k} \otimes e_\alpha.$$

We use "abuse of notation":

Define the "connection form" $A = A_i^\alpha{}_\beta dx^i$.

[Not a global 1-form, only on a local frame.]

Also define the WEDGE PRODUCT on $E \otimes E^*$ -valued forms:

$$D_A s = ds + A \wedge s \quad (s \in \Omega^k(E))$$

$$D_A \omega = d\omega + A \wedge \omega + (-1)^{k+1} \omega \wedge A \quad (\omega \in \Omega^k(E \otimes E^*))$$

Define the curvature $F_A = (D_A)^2$. (operator)
 check with $(D_A)^2 s = [D_i, D_j] s$ 10
 Form notation: $D_A s = D_A(ds + A s) = [D_i, D_j] s$

$$\begin{aligned} (d(dx \wedge \omega)) &= d^2 s + dA s - A d s \\ &= -d^2 s + A \wedge A s. \\ &= (dA + A \wedge A) s. \end{aligned}$$

Evidently

$$\boxed{F_A = dA + A \wedge A} \quad (\text{Cartan}).$$

is a 0th order operator $\mathcal{R}^0(E) \rightarrow \mathcal{R}^2(E)$

$\Rightarrow$ tensor in $\mathcal{R}^2(E)$! [you can check but already proven]

it has components

$$F_A = \frac{1}{2} (F_{ij}) dx^i \wedge dx^j$$

$$\boxed{F_{ij} = \partial_i A_{j\beta} - \partial_j A_{i\beta} + A_{i\gamma} A_{j\beta}^\gamma - A_{j\gamma} A_{i\beta}^\gamma}$$

$= [D_i, D_j]$

We can also prove the 2nd Bianchi identity:

$$\begin{aligned} D_A F_A &= d(dA + A \wedge A) + A \wedge (dA + A \wedge A) - (dA + A \wedge A) \wedge A \\ &= d^2 A + dA \wedge A - A dA \\ &\quad + A \wedge dA + A \wedge A \wedge A \\ &\quad - dA \wedge A - A \wedge A \wedge A \\ &= 0. \end{aligned}$$

Q. What is the meaning of curvature?

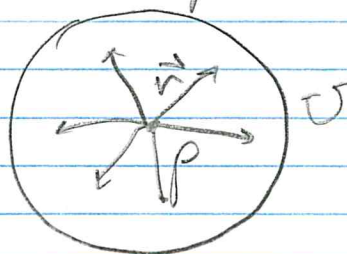
A. Holonomy = parallel transport around a closed loop.



Fix $p \in M$ and choose the gauge for $\nabla \otimes \rho$ by parallel transport along geodesics.

$$\nabla_{\vec{r}} \rho_\alpha = 0, \alpha = 1, \dots, n.$$

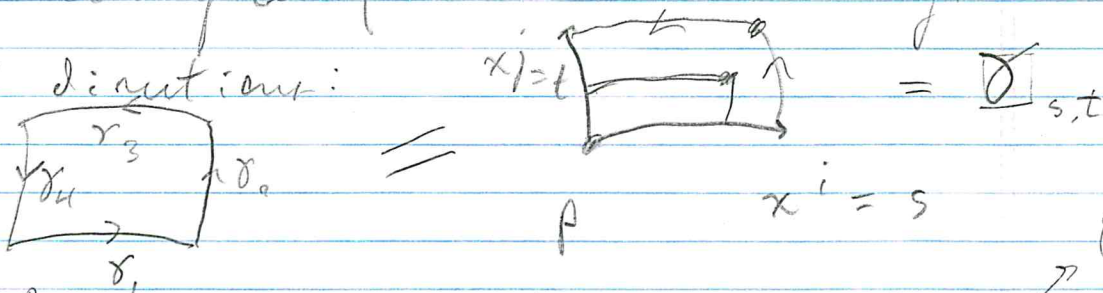
"Radial gauge."



$$\Rightarrow A_{i\beta}^\alpha(p) = 0!$$

Choose $i, j \in \{1, \dots, n\}$ and consider the rectangles of side s, t along the

i, j directions:



Taylor expansion:

$$A_{i,j} = 0 + s \cdot \partial_i A_j + t \partial_j A_i + O(2)$$

$$A_j = s \partial_i A_j + O(2).$$

On γ_1 : $A_i \equiv 0 \Rightarrow T_{\gamma_1} = 1$.

12.

On γ_2 : $A_j = s \overbrace{2_j A_j(0)}^{\text{constant!}} + O(2)$

$$\Rightarrow T_{\gamma_2} T_{\gamma_1} = (1 + t \cdot (s 2_j A_j)) + O(3).$$

On γ_3 : $A_i = t 2_j A_i(0) + O(2)$

$$\Rightarrow T_{\gamma_3} T_{\gamma_1} T_{\gamma_2}$$

$$= (1 - s \cdot t 2_j A_i(0)) \cdot (1 + t s 2_j A_j(0)) + O(3)$$

$$= 1 + s t (2_j A_i(0) - 2_j A_i(0)) + O(3)$$

On γ_4 : $A_i \equiv 0 \Rightarrow T_{\gamma_4} = 1$.

We conclude: $T_{\gamma_{1,2,3,4}} = 1 + s t F_{ij}(0) + O(3)$

F_{ij}

conversely: Theorem: Assume $F_A \equiv 0$ on a coordinate chart U . Then there exists a local frame in which $A \equiv 0$ on U .

Pf: Polar coords: $A_r \equiv 0$



$$0 = F_r \theta = 2_r A_\theta - 2_\theta A_r + [A_r, A_\theta]$$

$$\Rightarrow 2_r A_\theta = 0. \quad \text{But } A_\theta \rightarrow 0 \text{ as } r \rightarrow \infty$$

$$\Rightarrow A_\theta \equiv 0 \text{ also. } \square$$

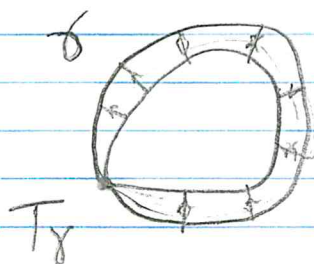
Note: in this frame, the transition functions are constant! $da_i' \equiv 0$.

Globally: [V. interesting that a connection can be nontrivial even if $A \equiv 0$ in every local frame!]

{ flat connections on M }

$\rightsquigarrow$ { representations of $\pi_1(M)$ }

[quotients!]



$$[X] \mapsto T_X \in \mathcal{M}^{n \times n}$$

well-defined on homotopy classes.

Q. What is the best connection on a vector bundle E ?

[A. One with the least amount of curvature.]